CNCM Online Round 2

CNCM Administration

August 8, 2020

2 Problems

Problem 1. Adi the Baller is shooting hoops, and makes a shot with probability p. He keeps shooting hoops until he misses. The value of p that maximizes the chance that he makes between 35 and 69 (inclusive) buckets can be expressed as $\frac{1}{\sqrt[b]{a}}$ for a prime a and positive integer b. Find $a+b$.

Problem 2. There is a rectangle $ABCD$ such that $AB = 12$ and $BC = 7$. E and F lie on sides AB and CD respectively such that $\frac{AE}{EB} = 1$ and $\frac{CF}{FD} = \frac{1}{2}$. Call X the intersection of AF and DE. What is the area of pentagon $BCFXE$?

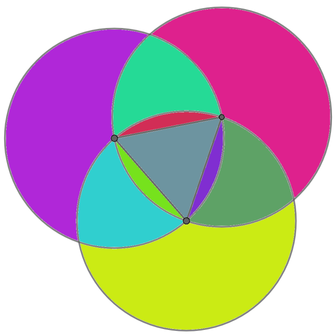
Problem 3. An ordered pair (n, p) is *juicy* if $n^2 \equiv 1 \pmod{p^2}$ and $n \equiv -1 \pmod{p}$ for positive integer n and odd prime p. How many juicy pairs exist such that $n, p \leq 200$?

Problem 4. On a chessboard with 6 rows and 9 columns, the Slow Rook is placed in the bottom-left corner and the Blind King is placed on the top-left corner. Then, 8 Sleeping Pawns are placed such that no two Sleeping Pawns are in the same column, no Sleeping Pawn shares a row with the Slow Rook or the Blind King, and no Sleeping Pawn is in the rightmost column. The Slow Rook can move vertically or horizontally 1 tile at a time, the Slow Rook cannot move into any tile containing a Sleeping Pawn, and the Slow Rook takes the shortest path to reach the Blind King. How many ways are there to place the Sleeping Pawns such that the Slow Rook moves exactly 15 tiles to get to the space containing the Blind King?

Problem 5. Consider a regular n-gon of side length 1. For each of its vertices, a circle of radius one is drawn centered at that vertex. The resulting figure, consisting of the polygon and the n circles, partitions the plane into $f(n)$ finite, bounded regions. Find

$$\sum_{n=3}^{25} f(n).$$

The first term corresponding to $i = 3$ is shown; each of the various colors corresponds to a distinct region with $f(3) = 10$. Note that the lines corresponding to the polygon are treated no differently than the arcs corresponding to the circles in counting regions.



Problem 6. Let S be the set of all ordered pairs (x, y) of integer solutions to the equation

$$6x^2 + y^2 + 6x = 3xy + 6y + x^2y.$$

S contains a unique ordered pair (a, b) with a maximal value of b. Compute $a + b$.

Problem 7. A circle is centered at point O in the plane. Distinct pairs of points A, B and C, D are diametrically opposite on this circle. Point P is chosen on line segment AD such that line BP hits the circle again at M and line AC at X such that M is the midpoint of PX. Now, the point $Y \neq X$ is taken for $BX = BY, CD \parallel XY$. IF $\angle PYB = 10^\circ$, find the measure of $\angle XCM$.